One-dimensional minimal fillings
with negative edge weights

A. O. Ivanov   Z. N. Ovsyannikov   N. P. Strelkova
A. A. Tuzhilin

Abstract

Ivanov and Tuzhilin started an investigation of a particular case of Gromov Minimal Fillings problem (generalized to the case of stratified manifolds). Weighted graphs with non-negative weight function were used as minimal fillings of finite metric spaces. In the present paper we introduce generalized minimal fillings, i.e. minimal fillings where the weight function is not necessarily non-negative. We prove that for any finite metric space its minimal filling has the minimum weight in the class of all generalized fillings of the space.

Key words: minimal filling, finite metric space.

1   Introduction

The problem concerning minimal fillings of finite metric spaces was posed by Ivanov and Tuzhilin in [1]. This problem arose in connection with two classical problems, the Steiner tree problem and the problem on minimal fillings of smooth manifolds (posed by M. Gromov, see [2]).

The objective is to find a weighted graph of minimum weight among all weighted graphs joining the points of a given finite metric space provided that for any two points in the metric space the distance between them is not greater than the weight of the shortest path connecting them in the graph. Now we give the precise definitions from [1].

Let $\mathcal{M} = (M, \rho)$ be a finite pseudometric space (a pseudometric space is a generalized metric space in which the distance between two distinct points can be zero), let $G = (V, E)$ be a connected graph joining M (by definition a connected graph G *joins* M if $M \subset V$), let $w : E \to \mathbb{R}_+$ be a function from the graph edge set to the set of non-negative real numbers, which is called the *weight function* of the *weighted graph* $\mathcal{G} = (G, w)$. The sum $\sum_{e \in E} w(e)$ of all edge weights of $\mathcal{G}$ is called *the weight of the graph* and is denoted by $w(\mathcal{G})$.

We define a distance function d_w on the set V by saying that the distance between two points is equal to the weight of a shortest path connecting these two points in the graph.

The weighted graph $\mathcal{G} = (G, w)$ is called a *filling* of the space $\mathcal{M}$ if for any two points $p, q \in M$ the condition $\rho(p, q) \leqslant d_w(p, q)$ holds. In this case the graph G is called the *type* of this filling. The number $\mathrm{mf}(\mathcal{M}) = \inf w(\mathcal{G})$, where infimum is taken over all fillings $\mathcal{G}$ of the space $\mathcal{M}$ is called the *minimal filling weight*, while a filling $\mathcal{G}$ provided $w(\mathcal{G}) = \mathrm{mf}(\mathcal{M})$ is called a *minimal filling* of the space $\mathcal{M}$.

In paper [1], only weighted graphs with non-negative weight function were used as fillings. In the present paper, we don't forbid negative weight edges and are looking for the minimum graph weight among fillings with arbitrary, not necessarily non-negative weight function. Such fillings will be called *generalized*.

The generalized minimal fillings have most of the properties proved in [1] for non-generalized minimal fillings. Some properties will be proved in section 3.

The main result of the present paper is the surprising fact that for any metric space its minimal filling weight (i.e. the minimum of weight on the set of fillings with non-negative weight function) equals the minimum of weight on the larger set of generalized fillings (i.e. fillings with arbitrary weight function).

This result simplifies the original problem of finding the minimal filling weight since now one does not have to verify whether the edge weights are non-negative. Using this result, A. Yu. Eremin in [3] has proved a formula for calculating minimal filling weight of a finite metric space.

2 Definitions

We have already given the definitions of a filling and a minimal filling of a finite pseudometric space in the Introduction.

Let a graph $G = (V, E)$ and an embedding $M \subset V$ be fixed. Consider $\inf w(\mathcal{G})$ taken over all the fillings $\mathcal{G}$ of M having the fixed type G. This infimum is called the *minimal parametric filling weight of the fixed type G* and is denoted by $\mathrm{mpf}(\mathcal{M}, G)$, while any filling of type G on which the infimum is reached is called a *minimal parametric filling of type G*, see [1].

So-defined minimal and minimal parametric fillings exist for any finite pseudometric space ([1, theorem 2.1]).

In a graph G joining the given finite set M, the vertices that correspond to the points from M will be called *boundary*, and the remaining vertices in the graph will be called *interior*. The graph edge connecting vertices u and v will be denoted by uv.

Now we modify the definitions of fillings given above by allowing the edge weights be negative.

Let G be a graph and $w\colon E \to \mathbb{R}$ an arbitrary function. Then the pair $(G, w) = (V, E, w)$ is called a *generalized weighted graph*. We define $d_w\colon V \times V \to \mathbb{R}$ by saying that $d_w(u, v)$ is the smallest among weights of simple (i.e., non-self-intersecting) paths connecting u and v. The function d_w is not necessarily non-negative and may violate the triangle inequality.

A generalized weighted graph $\mathcal{G}$ joining M is called a *generalized filling* of a finite pseudometric space $\mathcal{M} = (M, \rho)$ if for any $u, v \in M$ holds: $\rho(u, v) \leqslant d_w(u, v)$.

Fix a graph $G = (V, E)$ and an embedding $M \subset V$. The value $\inf w(\mathcal{G})$, where the infimum is taken over all generalized fillings $\mathcal{G}$ of $\mathcal{M}$ of the fixed type G is called the *generalized parametric minimal filling weight of type G* and is denoted by $\mathrm{mpf}_-(\mathcal{M}, G)$. A generalized filling $\mathcal{G}$ of type G is called a *generalized parametric minimal filling of type G* if $w(\mathcal{G}) = \mathrm{mpf}_-(\mathcal{M}, G)$.

The value $\inf \mathrm{mpf}_-(\mathcal{M}, G)$, where the infimum is taken over all trees G joining M and satisfying the condition that any vertex $v \in G$ of degree 1 is boundary (i.e., $v \in M$) is called the *generalized minimal filling weight of type G* and is denoted by $\mathrm{mf}_-(\mathcal{M})$. A generalized filling $\mathcal{G}$ is called a *generalized minimal filling of type G* if $w(\mathcal{G}) = \mathrm{mf}_-(\mathcal{M})$.

Remark 1. There is a similar definition for non-generalized minimal fillings: $\mathrm{mf}(\mathcal{M}) = \inf \mathrm{mpf}(\mathcal{M}, G)$ where there is no difference whether to take the infimum over trees or over arbitrary graphs G joining M. The definition in both cases is equivalent to the definition of minimal filling weight given above, see [1]. But the situation is different in the case of generalized minimal fillings — see example 1.

Remark 2. One certainly obtains an equivalent definition for generalized minimal filling if one takes $\inf \mathrm{mpf}_-(\mathcal{M}, G)$ only over trees without interior vertices of degree 2.

Remark 3. Suppose the embedding $M \subset V$ is organized in a such way that there is an interior vertex of degree 1 in G. We claim that then $\mathrm{mpf}_-(\mathcal{M}, G) = -\infty$. This easily follows from the fact that there are no restrictions on the

weight of the edge incident to this vertex and so the weight of this edge can be made any negative number regardless of what other edge weights are.

Example 1. We give an example where $\mathrm{mpf}_-(\mathcal{M}, G) = -\infty$ though for any edge in the graph there exists a path with boundary endpoints that goes through this edge (in contradiction to the cases like one considered above in which there is an interior vertex of degree 1). Consider a metric space consisting of two points $M = \{A, B\}$ with distance $\rho(A, B) = 1$. The figure shows the graph G joining M and the edge weights w. We assume $c > 0$.
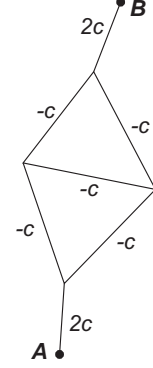
Now we have $d_w(A, A) = 0, d_w(A, B) = c, d_w(B, B) = 0$. Therefore for any $c \geqslant 1$ the graph (G, w) is a generalized filling for space $\mathcal{M}$. But $w(G) = -c$. A similar example can be easily constructed for any finite metric space.

Example 2. We give an example of a space $\mathcal{M}$ and a tree G with the generalized parametric minimal filling weight strictly less than the parametric minimal filling weight: $\mathrm{mpf}(\mathcal{M}, G) > \mathrm{mpf}_-(\mathcal{M}, G)$.

Consider $M = \{a, b, c, d\}$, $\rho(a, b) = \rho(c, d) = 4$, $\rho(b, c) = \rho(a, d) = \rho(a, c) = \rho(b, d) = 3$. Define graph $G = (V, E)$ as follows: $V = \{a, b, c, d, u, v\}$, $E = \{au, bu, uv, cv, dv\}$. We claim that in this case the minimal parametric filling weight is 8 (the reason is that $w(au) + w(ub) \geqslant \rho(a, b) = 4, w(cv) + w(vd) \geqslant \rho(c, d) = 4, w(uv) \geqslant 0$ so $\mathrm{mpf}(\mathcal{M}, G) \geqslant 8$ and 8 is the weight of the filling with the weight function $w(uv) = 0$, $w(e) = 2$ for other edges). On the other hand, it can be easily checked that the generalized parametric minimal filling weight is the weight of the filling with weights $w(uv) = -1$, $w(e) = 2$ for other edges. Thus we have an example where $\mathrm{mpf}(\mathcal{M}, G) = 8 > 7 = \mathrm{mpf}_-(\mathcal{M}, G)$.

Clearly, every generalized filling is a filling. So $\mathrm{mpf}_-(\mathcal{M}, G) \leqslant \mathrm{mpf}(\mathcal{M}, G)$ for any G and also $\mathrm{mf}_-(\mathcal{M}) \leqslant \mathrm{mf}(\mathcal{M})$.

The main result of this paper is the following theorem, which is proved in section 4.

Theorem. *Let $\mathcal{M}$ be a finite pseudometric space. Then $\mathrm{mf}_-(\mathcal{M}) = \mathrm{mf}(\mathcal{M})$.*

3   Properties of generalized minimal fillings

We assume throughout this section that the type of any generalized filling under consideration is a tree and all its vertices of degree 1 or 2 are boundary (i.e. the vertices are in M).

4

A *cyclic order* on a finite set M with n elements is an arbitrary ordering of its elements into a list or in other words a bijection $\pi\colon \mathbb{Z}_n \to M$. Consider a tree G joining M. For a given cyclic order π on M consider the paths in G connecting $\pi(k)$ and $\pi(k+1)$ where $k = 0, \ldots, n-1$. The collection of these paths is called a *tour through G corresponding to the cyclic order π*. A cyclic order π on M is called *planar with respect to G* if any edge in G is contained in exactly two paths from the tour corresponding to the cyclic order. Note that in [1] the term "tour" was used as a synonym of the expression "planar order".

The following simple assertion will be useful for us.

Lemma 1 *Suppose M includes all vertices of degree 1 from G. Then there exists a cyclic order on M that is planar with respect to G.*

Remark 4. To construct such an order, one can embed the tree G into the plane in arbitrary way and make a walk around this tree in the plane and denote by $\pi(k-1)$ the vertex number k from M that one meets along the walk. The fact is that any planar order can be obtained by such procedure. See [1, section 7] for more details and definitions. We shall need only the definition of planar order and lemma 1.

Lemma 2 *For any pseudometric space $\mathcal{M}$ its generalized minimal filling weight is positive (excluding the trivial case of distance function being identically zero in $\mathcal{M}$ — in that case the generalized minimal filling weight is zero).*

Proof. Let $\mathcal{G}$ be a generalized filling for space $\mathcal{M}$ of type G where $G = (V, E)$ is a tree with no interior vertices of degree 1. Consider an arbitrary order π on M that is planar with respect to G. The definition of planar order implies that the sum of all edge weights contained in the paths from the tour corresponding to the planar order equals the doubled weight of the filling. But the definition of generalized filling asserts that the weight of each path is greater or equal to the distance between its endpoints in space $\mathcal{M}$. Therefore:

$$w(\mathcal{G}) = \frac{1}{2}\sum_{k=0}^{n-1} d_w(\pi(k), \pi(k+1)) \geqslant \frac{1}{2}\sum_{k=0}^{n-1} \rho(\pi(k), \pi(k+1)) > 0 \ .$$

This completes the proof of lemma 2.

Lemma 3 *For any finite pseudometric space $\mathcal{M} = (M, \rho)$ and any graph $G = (V, E)$ joining M there exists a generalized minimal parametric filling of type G.*

5

Proof. A weight function w is determined by the string of its values assigned to the graph edges, i.e. by a vector from $\mathbb{R}^{|E|}$. The restrictions that distinguish the weight functions corresponding to generalized fillings (G, w) from other weight functions are linear non-strict inequalities. Therefore the subset Ω in $\mathbb{R}^{|E|}$ defined by these restrictions is an intersection of finitely many closed half-spaces. The linear function $w(\mathcal{G}) = x_1 + \ldots + x_{|E|}$ is bounded below on Ω by the assertion of lemma 2. Therefore this function attains its minimum on Ω. This completes the proof.

A filling will be said to be a *binary tree* if the type of the filling is a tree with all vertices of degree 1 or 3, moreover, with the property that all its vertices of degree 1 are boundary.

Let a subset $F \subset E$ of the edge set of a graph $G = (V, E)$ be fixed. Denote by $G_i = (V_i, F_i), i = 1, \ldots, m$ the connected components of the graph (V, F). Consider the new graph $G_F = (V_F, E_F)$ where $V_F = \{V_1, \ldots, V_m\}$ and two vertices V_i and V_j are adjacent if and only if there exists an edge $v_i v_j \in E \setminus F$ such that $v_i \in V_i$ and $v_j \in V_j$. The graph G_F is said to be the quotient graph of G by the set F.

The operation inverse to taking a quotient is called *splitting*. The graph G_F is said to be *obtained* from the graph G *by splitting the vertex* V_i if G_i is the only component consisting of more than one vertex. See [1] for related definitions and properties of such operations.

Lemma 4 *For any finite pseudometric space* $\mathcal{M} = (M, \rho)$ *there exits a minimal filling. Moreover, there exists a binary tree minimal filling.*

Proof. While searching for the minimal filling weight one can take the infimum $\inf \operatorname{mpf}_-(\mathcal{M}, G)$ only over trees G with all vertices of degree 1 and 2 lying in M (see remark 2). The set of trees with no more than $|M|$ vertices of degree 1 and 2 is obviously finite (here $|M|$ denotes the number of elements in M). So there is only a finite number of possible types for a filling of the space $\mathcal{M}$ and for each type there exists a generalized minimal parametric filling by the previous lemma. Now we obtain a generalized minimal filling of the space $\mathcal{M}$ by choosing the filling of minimal weight from this finite collection of generalized minimal parametric fillings.

Suppose the filling has some vertices of degree different from 1 and 3. Then we split each of the vertices by adding edges of zero weight. Thus we obtain a binary tree. This operation does not change the graph weight and the path lengths, so the resulting generalized filling is still minimal. This completes the proof.

Let $\mathcal{G}$ be a generalized filling of the space $\mathcal{M}$. A path γ in $\mathcal{G}$ connecting points x и y is called *exact* if $x, y \in M$ and its weight is equal to the distance between its endpoints in the space $\mathcal{M}$, i.e. $w(\gamma) = \rho(x,y)$. We denote by $\deg(v)$ the degree of the vertex v, i.e. the number of edges incident to it.

Lemma 5
1. *For any edge in a generalized minimal parametric filling there exists an exact path containing the edge.*
2. *For any two adjacent edges in a generalized minimal filling there exists an exact path containing the edge.*
3. *For any pair of edges incident to an interior vertex of degree 3 in a generalized minimal parametric filling there exists an exact path containing the pair of edges.*
4. *Fix an interior vertex v in a generalized minimal parametric filling and a subset of m edges incident to the vertex. Suppose $m > \frac{1}{2}\deg(v)$. Then there exists an exact path containing a pair of edges from the subset.*

Proof. 1. The first part is clear since otherwise we could decrease the edge weight and obtain a filling with smaller weight.

2. Suppose there are no exact paths containing the two adjacent edges xv and vy in a generalized filling $\mathcal{G} = (V, E, w)$. Consider the graph G' with one additional vertex: $V' = V \cup \{u\}$ and with the edge set $E' = E \backslash \{vx, vy\} \cup \{ux, uy, uv\}$. Define $w'(uv) = \varepsilon$, $w'(xu) = w(xv) - \varepsilon$, $w'(yu) = w(yv) - \varepsilon$, $w'(e) = w(e)$ for other edges. Then the lengths of all the paths containing the two edges xv and vy are reduced by 2ε and the lengths of other paths with boundary endpoints remain unchanged. Since no exact paths contained the two edges xv and vy, $\varepsilon > 0$ can be chosen in such a way that (G', w') is a filling. But $w'(\mathcal{G}') = w(\mathcal{G}) - \varepsilon$, which contradicts minimality of the original filling $\mathcal{G}$.

3. Follows from **4** by putting $m = 2, \deg(v) = 3$.

4. Suppose there is a set of $m > \frac{1}{2}\deg(v)$ edges incident to an interior vertex v provided that no two edges from the set are contained in an exact path. Define the weight function w' on G by reducing the weight of every edge from the set by ε and by adding ε to the weight of every other edge incident to v, while the remaining edge weights are left unchanged. By the assumption $m > \deg(v) - m$, so for any $\varepsilon > 0$ the weight of the graph decreases. On the other hand, for any sufficiently small $\varepsilon > 0$ the graph (G, w') is a filling. This contradiction completes the proof.

Remark 5. Each of the last three lemmas is a straightforward generalization of the corresponding result in the case of non-generalized fillings, see [1].

Moreover, the proof in the case of generalized fillings appears to be simpler since one does not have to check if the edge weights are non-negative.

Lemma 6 *Consider a boundary vertex in a generalized minimal parametric filling and consider an edge incident to the vertex. Then the edge weight is non-negative.*

Proof. Let p be a boundary vertex and px be an edge incident to it. Suppose x is boundary. Then $w(px) \geqslant \rho(p,x) \geqslant 0$, and the proof is finished. In the converse case, the degree of vertex x is not less than 3 and by lemma 5, there exists an exact path containing a pair of edges incident to x and different from the edge px. Denote the endpoints of the path by q and r (then we denote the path by q-r). Consider the paths p-q and p-r. These two paths have only one common edge px. The path q-r consists of all the edges contained in paths p-r and p-q excluding the edge px. Therefore:

$$\rho(q,r) = d_w(q,r) = d_w(q,p) + d_w(r,p) - 2w(px) \geqslant \rho(q,p) + \rho(r,p) - 2w(px),$$

on the other hand, $\rho(q,p) + \rho(r,p) \geqslant \rho(q,r)$, so $w(px) \geqslant 0$. This completes the proof.

Remark 6. Suppose the finite space under consideration is a non-degenerate metric space, i.e. the strict inequalities $\rho(x,y) > 0$ and $\rho(x,y) + \rho(y,z) > \rho(x,z)$ hold for any $x \neq y \neq z \neq x$. Then the weight of any edge incident to a boundary vertex in a generalized minimal parametric filling of this space is strictly positive. This easily follows from the proof of lemma 5.

Modification. Let $\mathcal{G} = (V,E,w)$ be a generalized filling of space $\mathcal{M}$, and let X and Y be vertices of degree 3 connected by a negative weight edge: $w(XY) = -2e < 0$. Denote by A and B the vertices adjacent to X and different from Y and denote by C and D the vertices adjacent to Y and different from X. Denote $w(XA) = a$, $w(XB) = b$, $w(YC) = c$, $w(YD) = d$. We construct the modified tree $G' = (V',E')$ as follows.

The vertex set is the same: $V' = V$. The edge set is modified: $E' = E \backslash \{XB, YD\} \cup \{XD, YB\}$. Define the weight function: $w'(XY) = 2e$, $w'(XA) = a - e$, $w'(YB) = b - e$, $w'(YC) = c - e$, $w'(XD) = d - e$, $w' = w$ on other edges of G'.
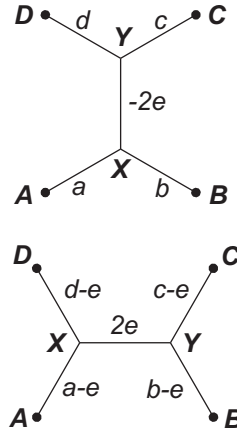
Lemma 7
1. *The modified graph $\mathcal{G}' = (G', w')$ is a generalized filling of $\mathcal{M}$ with the same weight $w'(\mathcal{G}') = w(G)$.*
2. *Suppose $u, v \in M$. Then $d_{w'}(u, v) \geqslant d_w(u, v)$.*
3. *Suppose a path in G with endpoints $u, v \in M$ contained the edge sequence AX, XY, YC. Then the path length has increased in G', i.e. $d_{w'}(u, v) > d_w(u, v)$.*

Proof. Statement **1** follows from **2**. The fact $w'(\mathcal{G}') = w(\mathcal{G})$ is obvious. Let us prove statements **2** и **3**. Consider a path with boundary endpoints. Suppose the path contains neither of the vertices X and Y. Then the path length obviously does not change after modification since the whole path remains unchanged. Suppose the path in G contained the edge sequence AX, XB (with weight $a + b$). Then the corresponding path in G' contains the edge sequence AX, XY, YB (with the same weight $a - e + 2e + b - e = a + b$). Therefore in this case the path length also remains unchanged after modification. The situation is similar in the cases where the path in G contained the edge sequence CY, YD, or BX, XY, YC, or AX, XY, YD.

Now suppose the path in G contained the edge sequence AX, XY, YC. Then the path length increases after modification since an arc of weight $a - 2e + d$ is replaced by an arc of weight $a - e + 2e + d - e = a + d$, which is strictly greater than $a - 2e + d$. The case of BX, XY, YD is similar. The proof is finished.

4   Proof of the main theorem

Let us recall the main theorem.

Theorem. *Let $\mathcal{M}$ be a finite pseudometric space. Then $\mathrm{mf}_-(\mathcal{M}) = \mathrm{mf}(\mathcal{M})$.*

Proof. As we have already mentioned above, it is obvious that $\mathrm{mf}_-(\mathcal{M}) \leqslant \mathrm{mf}(\mathcal{M})$. Hence it is sufficient to prove that there exists a filling with weight $\mathrm{mf}_-(\mathcal{M})$ and without negative edge weights.

Choose a filling $\mathcal{G}$ with the minimum number of exact paths among all the generalized minimal fillings of $\mathcal{M}$ that are binary trees (at least one such filling exists by lemma 4). Let us prove that there are no negative edge weights in filling $\mathcal{G}$ and thus G provides an example of non-generalized filling with weight $\mathrm{mf}_-(\mathcal{M})$.

Suppose there is an edge of negative weight. Denote it by XY. By lemma 5, there exists an exact path γ containing the edge XY. The vertices X and Y are interior since otherwise the weight of XY would be non-negative

by lemma 6. So we can modify the graph as described above. Moreover, by lemma 7 we can modify the graph in such a way that the weight of the path γ increases and the weights of other paths with boundary endpoints do not decrease. Since the modification preserves the graph weight the modified graph is a generalized minimal filling of $\mathcal{M}$. But the number of exact paths has reduced by at least 1 in the modified graph. This contradicts the choice of $\mathcal{G}$. The proof is finished.

Remark 7. The theorem does not hold if the function ρ on the boundary set M violates the triangle inequality. To be exact, only one of our lemmas uses the triangle inequality. This is lemma 6 on non-negativity of boundary edge weights.

Example 3. Consider the set $\{x, y, z\}$ and the function ρ: $\rho(x, y) = 1$, $\rho(y, z) = 2$, $\rho(z, x) = 5$. The triangle inequality does not hold but still one can search for the minimal filling weight. It is easy to see that in this case the generalized minimal filling weight and the minimal filling weight are different. They are 4 and 5 respectively. And there is a boundary edge of negative weight in the generalized minimal filling.

References

[1] Ivanov A. O, Tuzhilin A.A., One-dimensional M. Gromov's problem on minimal fillings, Matem. Sbornik, 2011, to appear, arXiv:1101.0106 [math.MG].

[2] Gromov M., Filling Riemannian manifolds, J. Diff. Geom. 1983. **18**, № 1. 1–147.

[3] Eremin A.Yu., Formula calculating the weight of minimal filling, Matem. Sbornik, 2011, to appear.